# A Study on Value-at-Risk and Lévy Processes

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**Abstract:** The Basle 2 Capital Accord issued by the Basle Committee on banking supervision has proposed a multiplier superior to 3 on banks' internal 99% 10-day Value-at-Risk calculated for market risk exposure. This ad hoc factor has not been fully explained and is poorly justified by arguing that the standard classical models of stock price dynamics do not adequately capture actual market risks. More generally, the current crisis has questioned a lot risk management practices about the determination of appropriate VaRs. In this paper, we revisit the computation of Value-at-Risk by introducing a new method based on Lévy processes. After a brief preliminary study where jump tests are performed in order to confirm the need for jump processes in financial modeling, we provide a new presentation of Variance Gamma Processes with Drift, that are reconstructed in an original way starting from the exponential distribution. Then, we display a fit of these processes on the American and French markets, before providing a new general Fourier formula that allows to compute VaR quickly and efficiently. Based on this formula, we conduct a study of the term structure of VaR, and provide a discussion of the Basle 2 and forthcoming Solvency 2 agreements.

**Keywords:** Value-at-Risk, Lévy processes, Variance Gamma processes, Fourier Transform, Basle 2, Solvency 2.

# Introduction

The Value-at-Risk (VaR) method is the most widely accepted method for risk measurement. It is used by all major financial institutions and is promoted by the Basle 2, Solvency 2 and UCITS 3 agreements. It is well known that the VaR metric is a function of a given probability distribution. A VaR measure is a procedure that, given a VaR metric, assigns values to portfolios. Choosing an appropriate VaR measure is an important and difficult task. There exists a wide range of studies about the development of VaR and risk measurement methods (see Jorion [2007]). The main purpose of this study is to put forward a new probabilistic framework for calculating VaR. Instead of using the classical extreme value theory, which applies only to tails and is not able to render (or be calibrated on) the center of distributions, we develop our method on the use of Lévy processes. The advantage of the complete distribution given by this alternative framework is to escape from the statistical estimation limitations due to the EVT framework. This enables to operate calculations with more confidence because of the ability to capture more accurately the true skewness and kurtosis of marginal distributions. Despite these advantages, it is difficult to calculate a VaR with Lévy processes with usual numerical methods. An important contribution of the paper is a new and fast general Fourier formula for calculating Lévy-VaR quickly and efficiently.

As far as the choice of Lévy process is concerned, our study is grounded on the use of Variance Gamma processes. These dynamics, together with similar processes, have been used extensively at the turn of the century for the reconstruction of volatility surfaces. The reader can refer to the well-known papers of Carr and Madan [1998], Carr, Chang and Madan [1998], or Carr, Geman, Madan and Yor [2002] for the application of Variance Gamma and CGMY processes in option pricing. In this paper, on the contrary, we concentrate on risk management, and conduct a study on the computation of VaR when asset return dynamics are assumed to follow Variance Gamma Processes with Drift (hereafter VGPD).

In heuristic terms, instead of starting from the Laplace second law (1778) of errors, namely the Gaussian distribution, our setting is built on the Laplace first law (1774) of errors, which states that the frequency of errors can be expressed as an exponential function of their numerical magnitude. Following several recent works (see for instance Kotz *et al.* [2001]), we designate by Laplace distribution the Laplace first law of errors. From there, we show how a Laplace motion can be constructed. It is this dynamics, which can be understood as a Brownian motion in a stochastic clock with gamma distribution, that has been named the variance gamma process by Madan *et al.* [1998]. Therefore, we give in this article a full and new reconstruction of the VGPD that relies on the exponential distribution.

The Basle 2 Capital Accord has proposed a multiplier superior to 3 on banks' internal 99% 10-day VaR calculated for market risk exposure. This ad hoc factor has not been fully explained and is poorly justified by arguing that the standard classical models of stock price dynamics do not adequately capture actual market risks. The objective of the Basle 2 agreement, with this recommendation of a protection equal to the maximum of the previous day's VaR and of 3 times the mean VaR observed during the past 60 days be enforced, is clearly to protect institutions against both quick and slow market evolutions, with the usual caveat that the past may not always be a good indicator of the future. A simplified rule states that financial institutions should post capital equal to 3 times the 99%-10 days VaR. According to this rule, an institution investing in the CAC40 Index, or in a well-diversified portfolio that mimicks the index's tracker, should protect itself by an order of magnitude of 35% of its investment amount. As far as the QIS4 of solvency 2 is concerned, it puts forward the 99.5%-1 year VaR, and promotes a 32% loading (for a duration of liabilities inferior to 3 years) for an investment in equities. We shall compare in this paper these percentages to those obtained when calibrating equity dynamics on VGPD processes and using the *ad hoc* new VaR formula.

The outline of the paper is as follows. In section 1, we conduct a preliminary study where jump tests are performed in order to confirm the need for jump processes in financial modeling. Section 2 provides a new presentation of VGPD, that are reconstructed in an original way starting from the exponential law. This section also displays a fit of these processes on the American and French markets. In section 3, we give a new general Fourier formula that allows to compute VaR quickly and efficiently. This formula, that we use with VGPD, could in fact be used with any other type of Lévy process (or more generally with any process for which we have an analytic characteristic function). Based on this formula, we conduct a study of the term structure of VaR, and provide a discussion of the Basle 2 and forthcoming Solvency 2 agreements.

# 1 Presence of jumps in the price dynamics

Before tackling the goal of this article, namely the computation of VaR when securities are modeled by a well-chosen class of geometric Lévy processes, we assess the relevance of modeling asset dynamics by jump processes. Indeed, one could argue that the kurtosis (so the leptokurticity) of asset dynamics can be recovered by using GARCH models, or, in continuous time, stochastic volatility: kurtosis is not necessarily related to the presence of jumps in dynamics. But, as we see below, real-world dynamics are best rendered by using jump processes. This is a strong motivation for using Lévy processes as building blocks for portfolio and risk management, without preventing the further addition of stochastic volatility type components.

## 1.1 Testing for the presence of jumps

To conduct this illustration, we rely on the statistical test constructed by Aït-Sahalia and Jacod [2008]. For sample values of the dynamics $X$, these authors define the estimator $\hat{H}$ as:

$$\hat{H}_2^{(4)}(t,\tau) = \frac{\sum_{k=1}^{\lfloor t/2\tau \rfloor} |X(k(2\tau)) - X((k-1)(2\tau))|^4}{\sum_{k=1}^{\lfloor t/\tau \rfloor} |X(k\tau) - X((k-1)\tau)|^4} = \frac{\sum_{k=1}^{\lfloor t/2\tau \rfloor} |\Delta X(t,2\tau)|^4}{\sum_{k=1}^{\lfloor t/\tau \rfloor} |\Delta X(t,\tau)|^4}$$

For a purely continuous process, the estimator converges to 2:

$$\hat{H}_2^{(4)}(t,\tau) \to 2$$

and, for a process that displays jumps, it converges to 1:

$$\hat{H}_2^{(4)}(t,\tau) \to 1$$

More precisely and because empirical dynamics are discrete and therefore discontinuous, the proper interpretation of this test is as follows. When the estimator $\hat{H}$ tends to 1, then the best continuous-time process for the representation of the tested dynamics is a jump process. Similarly, when the estimator $\hat{H}$ tends to 2, then the best continuous-time process for the representation of the tested dynamics is a continuous process.

## 1.2 Data and results

We tested many databases constituted of high-frequency quotes of American and French stocks. Invariably, the conclusion is that the best model for the representation of stock dynamics is a discontinuous model. For the sake of brevity, we only report the results obtained with a database of 784 630 consecutive quotes for Walmart (twelve days of market quotation), 628 970 consecutive quotes for Coca (twelve days of market quotation), 303 153 consecutive quotes for the French bank BNP (twenty days of market quotation), and 99 566 consecutive quotes for the French oil company Total (five days of market quotation).

| | Walmart | | | Coca Cola | | |
|---|---|---|---|---|---|---|
| Day | $\hat{H}$ | $T^c$ | $T^d$ | $\hat{H}$ | $T^c$ | $T^d$ |
| 18/05/2009 | 0.68* | 0.55 | 1.30 | 0.00 | 0.36 | 1.22 |
| 19/05/2009 | 1.40* | 1.18 | 1.45 | 0.64 | 0.67 | 1.27 |
| 20/05/2009 | 0.99* | 0.93 | 1.30 | 0.77 | 1.50 | 1.30 |
| 21/05/2009 | 0.98* | 0.40 | 1.29 | 0.83* | 0.36 | 1.16 |
| 22/05/2009 | 0.04 | 1.30 | 1.21 | 0.02 | 0.89 | 1.22 |
| 23/05/2009 | 0.24 | 1.36 | 1.15 | 0.44 | 1.37 | 1.15 |
| 26/05/2009 | 0.09 | 0.96 | 1.40 | 0.79 | 1.39 | 1.36 |
| 27/05/2009 | 0.56 | 0.67 | 1.27 | 0.10 | 1.16 | 1.29 |
| 28/05/2009 | 0.44 | 1.75 | 1.30 | 0.42 | 1.51 | 1.23 |
| 29/05/2009 | 0.49 | 1.50 | 1.21 | 1.08 | 1.63 | 1.26 |
| 30/05/2009 | 0.30 | 1.24 | 1.16 | 0.54 | 1.34 | 1.16 |
| 01/06/2009 | 0.21 | 1.15 | 1.25 | 0.19 | 1.44 | 1.35 |

Table 1: Testing for Jumps in the dynamics of Walmart and Coca Cola

In Tables 1 to 3, we give the value of the estimator $\hat{H}$, computed day by day using the high frequency data of these four companies. In the tables, we also represent the continuous and discontinuous thresholds $T^c$ and $T^d$ defining the 95% confidence intervals of the hypotheses "continuous dynamics" and "discontinuous dynamics". These thresholds admit the following meaning : if $\hat{H} < T^c$, there is only a 5% probability that the appropriate model be the continuous one (so that the probability of making a mistake by concluding that the best dynamics to represent the stock is a jump process is very small). Then, if $\hat{H} > T^d$, there is only a 5% probability that the appropriate model be the discontinuous one (so that the probability of making a mistake by concluding that the best dynamics to represent the stock is a continuous process is very small).

As can be observed in Tables 1 to 3, the indicator $\hat{H}$ admits values that are always very small, most often close or inferior to one. We added the superscript $^*$ when $\hat{H} > T^c$. We observe that even the few subscripted values of $\hat{H}$, for which it is theoretically not possible to conclude with a 95%

| Day | $\hat{H}$ | $T^c$ | $T^d$ |
|---|---|---|---|
| 21/01/09 | 0.55 | 1.31 | 1.26 |
| 20/01/09 | 0.86 | 1.69 | 1.32 |
| 19/01/09 | 0.98* | 0.79 | 1.40 |
| 16/01/09 | 0.80 | 1.45 | 1.34 |
| 15/01/09 | 0.97 | 1.19 | 1.36 |
| 14/01/09 | 1.44* | 0.54 | 1.39 |
| 13/01/09 | 1.01* | 0.64 | 1.27 |
| 12/01/09 | 1.12 | 1.78 | 1.24 |
| 09/01/09 | 0.93 | 1.63 | 1.30 |
| 08/01/09 | 0.95 | 1.46 | 1.34 |
| 07/01/09 | 0.96* | 0.67 | 1.47 |
| 06/01/09 | 0.94* | 0.92 | 1.49 |
| 05/01/09 | 0.79 | 1.56 | 1.28 |
| 02/01/09 | 1.05 | 1.19 | 1.32 |
| 31/12/08 | 0.91 | 0.92 | 1.35 |
| 30/12/08 | 1.03 | 1.50 | 1.34 |
| 29/12/08 | 1.01* | 0.63 | 1.30 |
| 24/12/08 | 1.08 | 1.68 | 1.24 |
| 23/12/08 | 0.99* | 0.43 | 1.31 |
| 22/12/08 | 0.95 | 1.01 | 1.48 |

Table 2: Testing for Jumps in the Dynamics of BNP

probability, are close to one. We insist that the tables presented in this article are representative of the results that are obtained in general with American or French stocks. These results altogether are enlightening and clear; they permit to reach the following conclusion: the best representation of stock dynamics is discontinuous. In the next section, we present our choice of discontinuous representation for stock dynamics and we derive a new formula for the computation of VaR.

# 2 A Lévy framework for asset returns

We have observed the importance of jumps in financial modeling. Let us now present a discontinuous framework well suited for the representation of asset dynamics. In a first step, we reconstruct the variance gamma process with drift from scratch and in an original way, namely building on the exponential

| Day | $\hat{H}$ | $T^c$ | $T^d$ |
|---|---|---|---|
| 26/01/09 | 1.03* | 0.88 | 1.47 |
| 27/01/09 | 1.06 | 1.89 | 1.23 |
| 28/01/09 | 0.74 | 1.23 | 1.47 |
| 29/01/09 | 0.69 | 1.60 | 1.35 |
| 30/01/09 | 0.96 | 1.69 | 1.32 |

Table 3: Testing for Jumps in the Dynamics of TOTAL

distribution. In a second step, we exhibit the results obtained by fitting these processes to actual data.

## 2.1 Main intuition and definitions

The main intuition to heuristically understand the nature of the Lévy framework, and consequently the resulting change and departure from the classical Gaussian framework of the financial literature, is the following. Instead of using the Laplace second law (1778) of errors (usually called the normal distribution or the Gaussian law), we choose the Laplace first law (1774) of errors, which states (in words) that the frequency of an error can be expressed as an exponential function of the numerical magnitude of the error disregarding sign. This contrasts the second law in which the frequency of an error is an exponential function of the square of the error. Following several recent works (see for instance Kotz *et al.* [2001]), we designate Laplace distribution the Laplace first law of errors, Laplacian noise the equivalent of Gaussian noise with Laplace distribution, and Laplace motion the stochastic process that plays the same role in the Laplacian domain as the Brownian motion does in the Gaussian one.

The correspondence with recent academic literature in finance is twofold. First, generalized Laplace distributions can be viewed as special cases of hyperbolic distributions, and this understanding connects the Laplace approach with the financial stream of hyperbolic and normal inverse Gaussian models for stock returns (see, e.g., Eberlein and Keller [1995], Barndorff-Nielsen [1997]). Second, Laplace motion is a special case of Lévy process and can be written as a Brownian motion time changed by a gamma process (clock time), and this view connects the Laplace approach with the financial literature related to time changed Brownian motion, i.e. Brownian motion evaluated at a random time (see, e.g., Geman *et al.* [2000]). Given the fact that Laplace motion can be understood as a Brownian motion in a stochastic clock with gamma distribution, this process has been named in seminal financial works of the nineties the variance gamma process (Madan *et al.* [1998]). Note that the Laplace framework can also be associated with the Kou [2000] model which contains both a continuous part modeled by a geometric Brownian motion and a discontinuous part where the logarithm of jumps admits a Laplace distribution.

We begin by presenting the symmetric Variance Gamma process introduced by Madan and Seneta [1990] to propose an alternative model for stocks quoted on the Sydney Stock Exchange, the goodness of fit of which has been examined for the statistical process of the log-price. A distinct issue is the one of the risk neutral densities because an enhancement of skewness is observed as a result of risk aversion in equilibrium, leading to the choice of an asymmetric Variance Gamma process (see Madan *et al.* [1998]).

We now provide a new presentation of Variance Gamma Processes with Drift, that are reconstructed in an original way starting from the exponential distribution. In particular, we aim to obtain the following VGPD characteristic function:

$$\Phi(u) = e^{i\theta t u} \left( \frac{1}{1 + \frac{1}{2}\sigma^2 v u^2 - i\delta v u} \right)^{\frac{t}{v}}$$

Let us start with the classical Laplace distribution whose probability distribution on $\mathbb{R}$ is given by the density function

$$f(x) = \frac{1}{2s} e^{-\frac{1}{s}|x-\theta|} \tag{1}$$

where $\theta \in \mathbb{R}$ and $s > 0$ are respectively location and scale parameters. Given the fact that the variance of (1) is $\sigma^2 = 2s^2$, $f$ can also be seen as a function of $\theta$ and $\sigma$:

$$f(x) = \frac{1}{\sigma\sqrt{2}} e^{-\frac{\sqrt{2}}{\sigma}|x-\theta|} \tag{2}$$

The characteristic function associated with $f$ is

$$\phi(u) = \frac{e^{i\theta u}}{1 + \frac{1}{2}\sigma^2 u^2} \tag{3}$$

A skewed generalization of the classical Laplace law can be designed by introducing a new parameter $\beta$ which controls the probability assigned to each side of $\theta$:

$$g(x) = \frac{\beta}{1+\beta^2} \frac{\sqrt{2}}{\sigma} \begin{cases} \exp\left(-\frac{\sqrt{2}\beta}{\sigma}(x-\theta)\right) & \text{if} \quad x \geq \theta \\ \exp\left(-\frac{\sqrt{2}}{\sigma\beta}(-x+\theta)\right) & \text{if} \quad x < \theta \end{cases} \tag{4}$$

Actually, when $\beta = 1$, the two probabilities are equal and the distribution is symmetric about $\theta$ : we obtain the classical Laplace law. An important property of $\beta$ is that the parameter is scale invariant.

It can be shown that the characteristic function associated with (4) is

$$\psi(u) = e^{i\theta u} \left( \frac{1}{1 - iu\frac{1}{\beta}\frac{\sigma}{\sqrt{2}}} \right) \left( \frac{1}{1 + iu\beta\frac{\sigma}{\sqrt{2}}} \right) = \frac{e^{i\theta u}}{1 + \frac{1}{2}\sigma^2 u^2 - i\delta u} \tag{5}$$

where $\delta$ is related to $\beta$ as

$$\delta = \left( \frac{1}{\beta} - \beta \right) \times \frac{\sigma}{\sqrt{2}} \tag{6}$$

Intuitively, $\delta$ represents the deviation from the mode $\theta$, and is related to the asymmetry of jumps. Recall that the parameter $\beta$ is scale invariant, whereas the scale of $\delta$ is given by the variance $\sigma$.

Based on the characteristic function (5), we can build the generalized asymmetric Laplace law as follows

$$\Psi(u) = (\psi(u))^t = \left( \frac{e^{i\theta u}}{1 + \frac{1}{2}\sigma^2 u^2 - i\delta u} \right)^t \tag{7}$$

whose corresponding density function can be written in terms of Bessel function of the third kind.

The characteristic function (7) can be expressed in the following manner

$$\Psi(u) = e^{i\theta t u} \left( \frac{1}{1 - iu\frac{1}{\beta}\frac{\sigma}{\sqrt{2}}} \right)^t \left( \frac{1}{1 + iu\beta\frac{\sigma}{\sqrt{2}}} \right)^t \tag{8}$$

which leads to the very useful representation

$$Y_t \overset{d}{=} \theta t + (G_1 - G_2) \tag{9}$$

where $G_1$ and $G_2$ are two i.i.d. Gamma $\mathcal{G}(t, \lambda_1)$ and $\mathcal{G}(t, \lambda_2)$ random variables with

$$\lambda_1 = \frac{\beta\sqrt{2}}{\sigma} \quad \text{and} \quad \lambda_2 = \frac{\sqrt{2}}{\sigma\beta}$$

and where a Gamma random variable $\mathcal{G}(t, \lambda)$ is defined by the density

$$f_{\mathcal{G}(t,\lambda)}(x) = \frac{\lambda^t}{\Gamma(t)} x^{t-1} e^{-\lambda x}$$

and spectrum

$$\Phi_{\mathcal{G}(t,\lambda)}(u) = \left(\frac{1}{1-\frac{iu}{\lambda}}\right)^t \tag{10}$$

Also note that, solving for (6), we have

$$\beta = \beta(\sigma,\delta) = -\frac{\delta}{\sigma\sqrt{2}} + \sqrt{1+\frac{\delta^2}{2\sigma^2}}$$

The process defined by (8) and (9), where $t$ is a time, is underparametrized: $\theta$, $\sigma$, and $\beta$ are not sufficient to permit any behavior of the four first moments of financial dynamics. We need to introduce a fourth parameter $v$, and we do this by performing a homothety on $\sigma$, $\delta$ and $t$ as follows:

$$\sigma \to \sigma\sqrt{v}$$

and

$$\delta \to \delta v$$

and

$$\theta \to \theta v$$

and

$$t \to \frac{t}{v}$$

thus defining $G_1' \rightsquigarrow \mathcal{G}(t, \frac{\hat{\beta}\sqrt{2}}{\sigma\sqrt{v}})$ and $G_2' \rightsquigarrow \mathcal{G}(t, \frac{\sqrt{2}}{\hat{\beta}\sigma\sqrt{v}})$ where $\hat{\beta} = \beta(\sigma\sqrt{v}, \delta v)$, and yielding finally the new random variable

$$X_t \overset{d}{=} \theta t + (G_1' - G_2') \tag{11}$$

which is the marginal r.v. at time $t$ of a variance gamma process with drift.

The characteristic function associated with (11) is

$$\Phi(u) = e^{i\theta t u}\left(\frac{1}{1-iu\frac{1}{\hat{\beta}}\frac{\sigma\sqrt{v}}{\sqrt{2}}}\right)^{\frac{t}{v}}\left(\frac{1}{1+iu\hat{\beta}\frac{\sigma\sqrt{v}}{\sqrt{2}}}\right)^{\frac{t}{v}} \tag{12}$$

Replacing $\hat{\beta}$ by its value and developing, we finally obtain the searched characteristic function

$$\Phi(u) = e^{i\theta t u}\left(\frac{1}{1+\frac{1}{2}\sigma^2 v u^2 - i\delta v u}\right)^{\frac{t}{v}} \tag{13}$$

where the right part of the right member of this equation is the well-known characteristic function of a variance gamma random variable, and the left part of this member accounts for an added pure drift.

In this article, we work with processes whose marginal variables $X_t$ are characterized as above in formula (13). The computation of the four first moments yields:

1. Mean
$$E\left(X_t\right) = \left(\delta + \theta\right) t$$
The mean depends both on the drift $\theta$ (mode of the Laplace distribution) and on the deviation due to asymmetrical jumps.

2. Variance
$$E\left(\left[X_t - E\left(X_t\right)\right]^2\right) = \left(\sigma^2 + \delta^2 v\right) t$$

3. Asymmetry
$$E\left(\left[X_t - E\left(X_t\right)\right]^3\right) = \left(2\delta^3 v^2 + 3\delta\sigma^2 v\right) t$$
Note that the sign of this expression is the same as the sign of $\delta$, motivating the introduction of $\theta$ to permit means and skewnesses of opposite signs.

4. Leptokurticity
$$E\left(\left[X_t - E\left(X_t\right)\right]^4\right) = (3\sigma^4 v + 12\sigma^2\delta^2 v^2 + 6\delta^4 v^3)t + (3\sigma^4 + 6\sigma^2\delta^2 v + 3\delta^4 v^2)t^2$$
Note that when $v = 0$, $E\left(\left[X_1 - E\left(X_1\right)\right]^4\right) = 3\sigma^4$ yielding a kurtosis coefficient $K = 3$.

These moments are interesting to mention for two reasons. First, they can be used as a tool to calibrate distribution parameters. Second, they are interesting *per se*, because they give clear meaning to the model parameters, which is always important in the real life. Indeed, lots of sophisticated models, including Lévy models, have been designed in finance and actuarial science, but not all of them can be easily used by practitioners due to the difficulty to interpret parameters (e.g. the well-known Heston model of stochastic volatility).

We conclude by a further motivation for the use of variance gamma processes with drift. When classical financial rests on the function $e^{-x^2}$, and therefore on exponentially decreasing tails, and when alternatives proposed in the sixties by Mandelbrot promote Paretian-like behaviors (tails decreasing like $\frac{C}{x^\alpha}$), fits of empirical distributions indicate that the reality should be between the two, namely that tails should be semi-heavy. Variance Gamma processes, or similar classes of Lévy processes, possess tails that behave asymptotically like $C\frac{e^{-\beta x}}{x^\alpha}$, providing therefore adequate tools for the modeling of empirical return distributions.

## 2.2 Data and results

We display here the results of the calibration of Variance Gamma Processes with Drift to real-world data. This study was conducted on French daily quotations. Again, for the sake of brevity, we report only a representative part of the overall results.

Table 4 displays the calibration by maximum likelihood of VGPD parameters on the returns of top French companies and of the French CAC40 Index over the period (01/03/01 - 04/15/09). It can be observed that the drift of returns, proportional to $\delta + \theta$ was either positive (e.g. Vinci) or negative (e.g. AXA) over the reference period. Also, the skewness sign of calibrated theoretical distributions, given by $\delta$, is predicted as often positive (e.g. BNP) as negative (e.g. Carrefour), in full consistence with what is observed in empirical distributions. These two features confirm the importance of introducing the parameter $\theta$, so of using VGPD rather than standard VG processes. We finally remark that the calibrated values of $\sigma$ and of $v$ (the latter parameter being the one that drives kurtosis) are pretty stable: in a daily fit of returns, the order of magnitude of $\sigma$ is 0.02 and the order of magnitude of $v$ is 1.

| Quote | $\delta$ | $\sigma$ | $v$ | $\theta$ |
|---|---|---|---|---|
| AXA | -0.0006 | 0.0290 | 1.1938 | 0 |
| BNP | -0.0011 | 0.0229 | 1.1519 | 0.0010 |
| Carrefour | 0.0006 | 0.0189 | 0.9549 | -0.0010 |
| PPR | 0 | 0.0235 | 1.3933 | -0.0006 |
| Total (Oil Company) | -0.0013 | 0.0173 | 0.7529 | 0.0013 |
| Vinci | 0.0003 | 0.0191 | 1.1516 | 0 |
| CAC40 | -0.0011 | 0.0154 | 0.9603 | 0.0008 |

Table 4: Calibrated Asymmetric Laplace Parameters

Figure 1 displays the fit of the daily returns of BNP by VGPD and Gaussian dynamics. The histogram represents the empirical returns. The curve

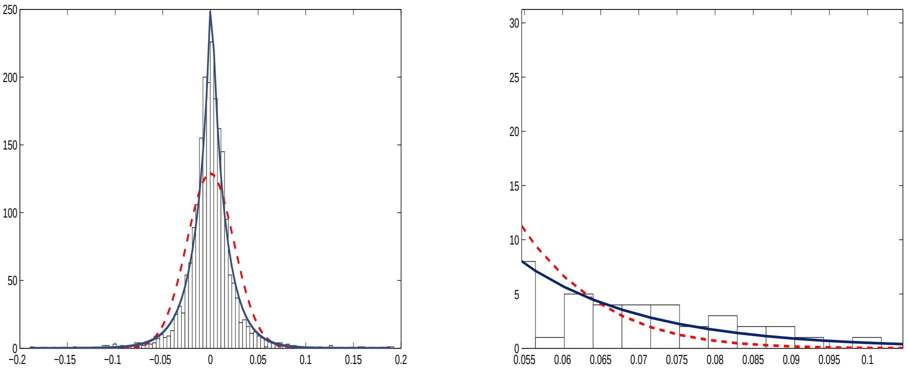


Figure 1: Fit and Tail Fit of BNP daily returns

in plain represents the VGPD return distribution fitted to the empirical dynamics by means of maximum likelihood (so using the parameters obtained as for the above Table 4). The dashed curve represents the Gaussian return distribution fitted to the empirical dynamics, again by means of maximum likelihood. The period of study is also (01/03/01 - 04/15/09).

As appears clearly from the left panel of Figure 1, the Gaussian model is particularly awkward in representing real-world return data. The peak of the dashed curve is as high as approximately 60% of the peak of the histogram: small positive or negative return values are highly underestimated. Similarly, we observe that medium return values are highly overestimated by the Gaussian model. As far as extreme values are considered, they are examined in the right panel of the same figure. The VGPD model performs well wherever the Gaussian model fails. Indeed, the VGPD model does not underestimate small return values and it does not overestimate medium return values. Also, it renders properly the asymmetry of real-world return data. Finally, and as the right panel of figure 1 illustrates, it fits very well empirical tails. To sum up, and as Figure 1 clearly shows, the VGPD model is well-suited for the representation of stock return dynamics.

Figure 1 (right) displays the fit of the right tail of BNP daily returns. This graph confirms that the Gaussian model is totally inappropriate for representing extreme return values: the Gaussian curve quickly touches zero and in fact does not take into account extreme return values. On the contrary, the VGPD model performs well in fitting empirical tails. The VGPD distribution curve renders properly the average level of the bars of the histogram.

Finally, we display in Figure 2 the distribution fit of the CAC 40 index

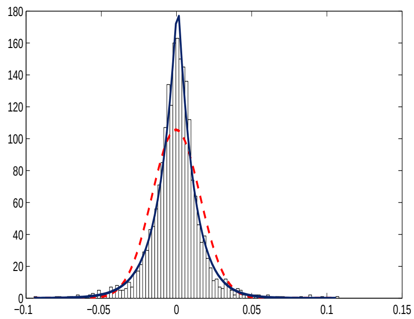


Figure 2: Fit of CAC40 daily returns

daily returns. Exactly as for stock returns, the conclusion is unambiguous: the fit of a Gaussian model can not be considered satisfactory and brings along a strongly biased and simplified vision of risk, whilst the fit of the AL models proves to be very consistent with empirical market data.

# 3 VaR with VGPD underlyings

We now come to the ultimate goal of this article: the computation of VaR when asset return dynamics are represented by Lévy processes. We start by giving a new formula that allows to compute VaR very efficiently. We then use this formula (which is valid for all types of Lévy processes) in the VGPD context to compute VaRs at various confidence levels and time horizon. Finally, we go the opposite way, computing implicit process parameters based on given VaR levels.

## 3.1 New computation of Lévy-VaR

We derive below a new formula allowing to compute directly the cumulative distribution associated with any Lévy process as a Fourier transform of a well behaved function of the characteristic function.

Recall that VaR is defined by

$$F\left(\mathrm{VaR}_{\alpha}\right) = \alpha$$

where $\alpha$ is the no-ruin probability and $F$ is the c.d.f. of the variable under study. We aim at achieving an easy computation of VaR with VGPD. In general, the c.d.f. $F$ and the density $f$ of an indefinitely divisible random variable are unknown, when its characteristic function $\Phi$ is known. A first naive algorithm to compute VaR could be $\Phi \rightarrow$ Inverse Fourier Transform $\rightarrow f$ and $f \rightarrow$ Quadrature $\rightarrow F$, plus a terminal root search on $F$ to obtain VaR.

Actually, one has the following formula

$$F(x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \frac{\Phi(u)}{iu} du$$

which allows to quicken matters, because it bypasses the use of the density. A second algorithm can then be designed as $\Phi \rightarrow$ Inverse Fourier Transform $\rightarrow F$, plus a terminal root search. However, the use of the above formula can be problematic, because it is not stable around $u = 0$. This is the reason why we design a new algorithm, with the formula

$$F(x) = \frac{e^{ax}}{2\pi} \int_{-\infty}^{+\infty} e^{iux} \, \frac{\Phi(ia - u)}{a + iu} \, du \tag{14}$$

valid with any positive real number $a$. The general approach remains identical: $\Phi \rightarrow$ Inverse Fourier Transform $\rightarrow F$, and a terminal root search on this $F$ to obtain VaR.

The formula (14) can be obtained as follows. Let $f$ be the density of the infinitely divisible distribution $X$ under study, and $\Phi$ its characteristic function which can be expressed as:

$$\Phi(u) = \int_{-\infty}^{+\infty} e^{iux} f(x) dx$$

First, $a$ being any strictly positive real number, we introduce the new function

$$g(x) = e^{-ax} F(x) = e^{-ax} \int_{-\infty}^{x} f(s) ds$$

The Fourier transform of this function is given by

$$\Lambda(u) = \int_{-\infty}^{+\infty} e^{iux} g(x) dx = \int_{-\infty}^{+\infty} e^{iux} \left( e^{-ax} \int_{-\infty}^{x} f(s) ds \right) dx$$

so that

$$\Lambda(u) = \int_{-\infty}^{+\infty} \int_{-\infty}^{x} e^{iux} e^{-ax} f(s) ds dx$$

Noting that $-\infty < s < x < +\infty$, we can swap the integrals as

$$\Lambda(u) = \int_{-\infty}^{+\infty} \int_{s}^{+\infty} e^{iux} e^{-ax} f(s) dx ds$$

Therefore

$$\Lambda(u) = \int_{-\infty}^{+\infty} f(s) \left( \int_{s}^{+\infty} e^{iux} e^{-ax} dx \right) ds = \int_{-\infty}^{+\infty} f(s) \left[ \frac{e^{-(a-iu)x}}{-(a-iu)} \right]_{s}^{+\infty} ds$$

and, because $|e^{-(a-iu)x}| = e^{-ax}$ tends to zero when $x$ becomes infinite

$$\Lambda(u) = \int_{-\infty}^{+\infty} f(s) \left( -\frac{e^{-(a-iu)s}}{-(a-iu)} \right) ds = \frac{1}{a-iu} \int_{-\infty}^{+\infty} f(s) e^{-(a-iu)s} ds$$

Finally

$$\Lambda(u) = \frac{1}{a-iu} \int_{-\infty}^{+\infty} f(s) e^{i(u+ia)s} ds$$

permits to write readily

$$\Lambda(u) = \frac{\Phi(u+ia)}{a-iu}$$

The function $g$ being the inverse Fourier transform of $\Lambda$

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \Lambda(u) du$$

we obtain

$$e^{-ax} F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \frac{\Phi(u+ia)}{a-iu} du$$

so that finally

$$F(x) = \frac{e^{ax}}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \frac{\Phi(u+ia)}{a-iu} du \tag{15}$$

A simple change of variable ($u = -u$) yields the alternative formula

$$F(x) = \frac{e^{ax}}{2\pi} \int_{-\infty}^{+\infty} e^{iux} \frac{\Phi(-u+ia)}{a+iu} du \tag{16}$$

One can use both formulas (15) or (16), in which a choice of characteristic function $\Phi$ has to be made, and perform an Fast Fourier Transform (FFT) to obtain $F$, before achieving a root search on $F$ to compute VaR. With Matlab, the result comes quasi-instantaneously because an FFT code applied to a vector of values of $\Phi$ yields quickly a vector of values of $F$, to which it remains to apply a simple search function.

## 3.2 VaR with VGPD

We compute the VaR attached to an investment in the French CAC40 Index. We calibrate the Index process on the data covering the period (01/03/01 - 04/15/09). The parameters of the VGPD representing the CAC 40 Index over that period are given in Table 4. Based on this VGPD dynamics, we compute VaR at various confidence intervals and protection time windows using formulas (13) and (14).

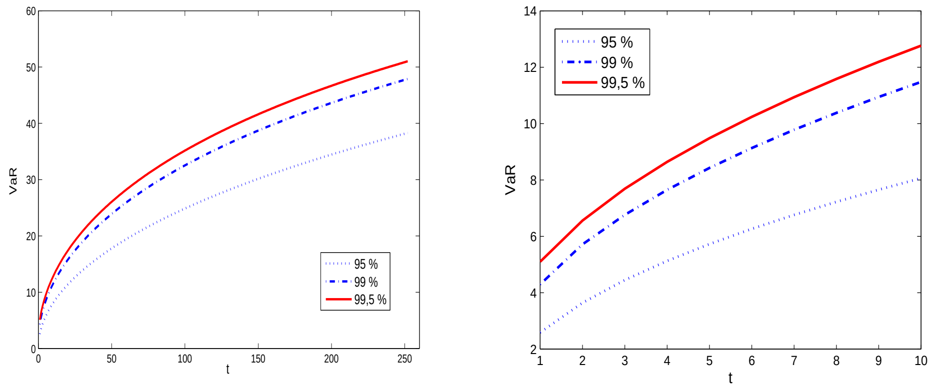


Figure 3: VaR w.r.t. investment horizon (in days)

In Figure 3 (left), we display VaR computed respectively at the 95, 99 and 99.5% confidence intervals, and for investment horizons ranging up to 252 open days, so up to one year. The following stylized features can be observed. First, and obviously, the higher the investment horizon, the higher the VaR, or equivalently the amount of the required protection. Second, and this is also obvious, the higher the confidence level, the higher the VaR. Third, when the confidence level is high, increasing it further does not change very much the amount of required protection. Fourth, a change in the confidence interval has a small impact on VaR compared to a change of the investment horizon. This is very important, because this sheds some light on many debates about whether a VaR at the 99 or the 99.5% confidence interval should be chosen. Clearly, this is not the most important point, and the real difficulty lies in the choice of investment horizon. Consider for instance a very illiquid corporate bond, held by a bank, not by an insurance company, so that the holding investment is supposed (by the regulator and by the bank itself) short, typically of the order of magnitude of a couple of days. With these instruments, it is not clear how many days are necessary for computing VaR, and the 30 open days VaR will be twice as big as the 10 open days one. In this instance also, the confidence level will not be critical for the determination of the appropriate protection. In Figure 3 (right), we plot a zoom and display VaR computed respectively at the 95, 99 and 99.5% confidence intervals, and for investment horizons ranging up to 10 open days. At short horizons also, we observe that the choice of investment horizon is far as important as the choice of confidence level. As far as the 99%-10 days VaR is concerned, it is important in finance because it it the one promoted by the Basle 2 Agreement. In the case of the CAC40 Index, it is equal to approximately 11.5% of the total investment.

As far as medium term horizons are concerned, the VGPD model predicts a 99.5%-1 year VaR equal to about 51% of the total investment. The difference with the 32% level promoted by the QIS4 of Solvency 2 is huge. This is a clear indication that a 32% loading does not correspond to a 99.5%-1 year VaR. Our conclusion stops here, and we do not go as far as to recommend a 51% loading. Why? Because, and as examined above, the sensitivity to the investment horizon is huge, and it remains to be shown that setting a one year horizon is the best and most relevant choice in the interest of insurance companies. Note finally that the 32% loading of QIS4 corresponds to a 99%-4 months VaR or to a 95%-9 months VaR on the French market (based on the VGPD model).

## 3.3 Implicit Processes

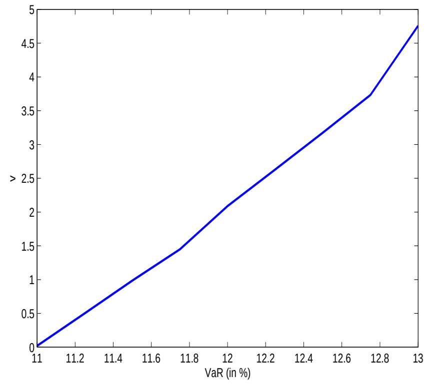


Figure 4: Implicit $v$ as a function of the $99\% - 10D$ VaR

In the previous subsection (see Figure 3), we compute the 99%-10 days VaR associated with the CAC40 Index. It is equal to $11,4763\%$. Assume now that VaR has been corrected by the risk manager, or computed based on another model, or just given as a rule of thumb and as the result of a consensus. Assume for instance that the 99%-10 days VaR has been estimated at various levels ranging between 11 and 13%. The question that arises then is to what Lévy process do these VaRs correspond to. To simplify matters, we assume all the parameters of the Lévy process describing the CAC40 Index fixed, except the kurtosis parameter $v$. In Figure 4, we plot the implicit $v$ corresponding to the chosen VaR level (performing a root search on the function given VaR, which is itself a root search on the function $F$ given in this article). Not surprisingly, the higher the VaR selected by the risk manager, the higher the kurtosis parameter of the VGPD process. This is an example of a mapping between VaR and kurtosis, when the litterature is more accustomed to direct matches between VaR and variance. This is also an example where we do the contrary of what is presented in the previous subsection. Here, we are not interested in VaR *per se*, but in implying the regulators or risk managers' view on the market from the VaR levels that they promote.

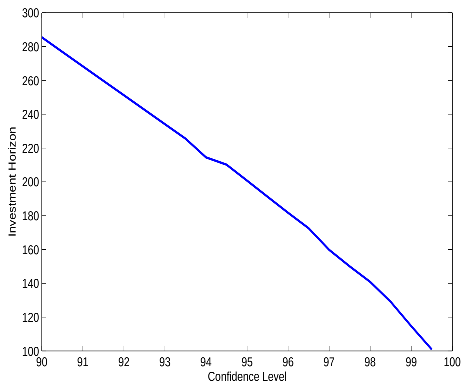


Figure 5: Implicit Basle 2 Curve

Another interesting experience consists in assuming that the Basle 2 agreement's spirit is the incitation to post capital equal to three times the 99%-10 days VaR, so to three times $11,4763\%$ in the case of the CAC40. The question is then to what type of VaR (on which a factor 1 and not 3 is imposed) does this loading correspond to? In Figure 5, we plot the implicit (confidence level, investment horizon) couples yielding a VaR equal to three times $11,4763\%$, so to $34,4289\%$. We see that with such a loading, an investment in the CAC40 is protected a hundred open days at the 99.5% confidence level. A larger investment horizon will correspond of course to a smaller confidence level. For instance, if the investment horizon is one year, then this investment can be considered protected only at a confidence level of about 92%.

# Conclusion

This article offers a new method for the computation of VaR when assets are modeled by Lévy processes. After a brief section which performs jump tests and motivates our study, we rebuild the Variance Gamma Process with Drift from the beginning, namely starting with the first law of Laplace - the exponential law. We provide a new formula that allows to compute VaR instantaneously when asset returns are modeled by Lévy processes. We also discuss the VaR levels promoted by the Basle and Solvency 2 agreements and compare them to those obtained with the Lévy method. The proposed algorithm is calibration efficient and stable, because it uses all the points of empirical distributions, contrary to the EVT method which relies on the calibration and modeling of a unique distribution tail.

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