

## Measuring Risk Dependencies in the Solvency II-Framework

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### *Abstract*

In April 2009 the European Parliament adopted a directive “on the taking-up and pursuit of the business of Insurance and Reinsurance” (Solvency II). According to this Solvency II directive the Solvency Capital Requirement (SCR) corresponds to the economic capital needed to limit the probability of ruin to 0.5 %. This implies that (re-)insurance undertakings will have to identify their overall loss distributions. In addition to the challenging duty of determining loss distributions for the single risks that insurance companies are facing, it will be probably very difficult to aggregate those loss distributions as well. The standard approach of the mentioned Solvency II directive proposes the use of a correlation matrix for the aggregation of the single so-called risk modules respectively sub-modules. In contrast to the standard approach the future regulations for internal models only determine that dependencies have to be recognized. Thereby the way of how insurers have to model these dependencies is not explicitly prescribed.

In our paper we will analyze the method of risk aggregation via the proposed application of correlations. We will find serious weaknesses, particularly concerning the recognition of extreme events, e. g. natural disasters, terrorist attacks etc. The reason for this is that correlations compress information about dependencies into a single ratio. Therefore important information concerning the tail of a distribution may possibly not be considered. In contrast, multivariate distribution functions provide full information with respect to dependencies between the relevant risks. However, aggregation of risks through “traditional” multivariate modeling causes technical difficulties. A possible solution for this dilemma can be seen in the application of copulas. Copulas allow the separate modeling of single loss distributions for each type of risk on the one hand and the dependencies between them on the other hand. Afterwards they can be combined by using the concept of copulas.

Therefore we come to the conclusion that it would have been desirable to fix the concept of copulas in the new solvency directive. Even though the concept of copulas is not explicitly mentioned in the directive, there is still a possibility of applying it. (Re-)insurers will be able to design their internal models by using an aggregation method more complex but even more precisely (e. g. copulas) than the solely utilization of a correlation matrix. It is clear that modeling dependencies with copulas would incur significant costs for smaller companies that might outbalance the resulting more precise picture of the risk situation of the insurer. However, incentives for those companies who use copulas, e. g. reduced solvency capital requirements compared to those who do not use it, could push the deployment of copulas in risk modeling in general.

### *Keywords*

coefficient of tail dependence, copulas, multivariate distribution function, Pearson’s linear correlation, risk aggregation, risk dependencies, risk measures, Solvency Capital Requirement, Solvency II directive, Solvency II, Spearman’s rank correlation

## 1. Introduction

The Solvency II directive focuses on an *economic risk-based approach*<sup>i</sup> and therefore obliges insurance undertakings to determine their overall loss distribution function. The increasing complexity of insurance products makes it necessary to consider dependencies between the single types of risk to determine this function properly. Neglecting those dependencies may have serious consequences underestimating the overall risk an insurer is facing. On the other hand, assuming complete dependency between risks may result in an overestimate of capital requirements and therefore incur too high capital costs for an insurance company. The Solvency II draft directive acknowledges this fact and proposes recognition of dependencies by the use of linear correlations. Reason is that correlations are relatively easy to understand and to apply. However, the use of correlation requires certain distributional assumptions which are invalidated f. ex. by non-linear derivative products and the typical skew and heavy tailed insurance claims data.<sup>ii</sup> Therefore aggregation of insurance risks via correlations may neglect important information concerning the tail of a distribution.

In contrast, copulas provide full information of dependencies between single risks. Therefore they have become popular in recent years. As they allow the separate modeling of risks and the dependencies between them, it will also be possible to explore the impact of (different) dependency structures on the required solvency capital if they are used.<sup>iii</sup> Different dependency structures can be modeled on the one hand through modified parameters of the copula function and on the other hand through the choice of a completely different copula family.

Our aim is to give an overview over the concept of copulas, to analyze and discuss their possibly application in the context of Solvency II and finally to make them accessible to a wider circle of users. In this context we would also like to discuss, if the new Solvency II directive forms an accurate concept for considering risk dependencies or if further adjustments should be made. Relating to that it will also be necessary to discuss dependency ratios like the correlation coefficient but also others (f. ex. Spearman's rank correlation).

We will therefore start with an overview over dependency ratios. In section 3 we will continue with the introduction of copulas and illustrate different families and types of copulas. After that we will briefly describe how copulas and multivariate distributions can be determined out of empirical data in section 4. The paper will continue with an assessment of the presented dependence concepts in section 5 and end with a description and an assessment of the consideration of risk dependencies in the Solvency II framework in section 6.

## 2. Dependency ratios

Using the linear correlation coefficient is a very rudimentary, but also simple way of describing risk dependencies in a single number. The linear correlation coefficient is defined by the following Formula 1:

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) * \text{Var}(Y)}},$$

$\rho(X, Y)$  is the linear correlation coefficient of  $X$  and  $Y$ ,  
 $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$  is the covariance of  $X$  and  $Y$  and  
 $\text{Var}(X)$  and  $\text{Var}(Y)$  are the finite variances of  $X$  and  $Y$ .

**Formula 1: Pearson's linear correlation coefficient**

In case of multiple dimensions the so called correlation matrix needs to be applied. Formula 2 shows this symmetric and positive semi-definite correlation matrix:

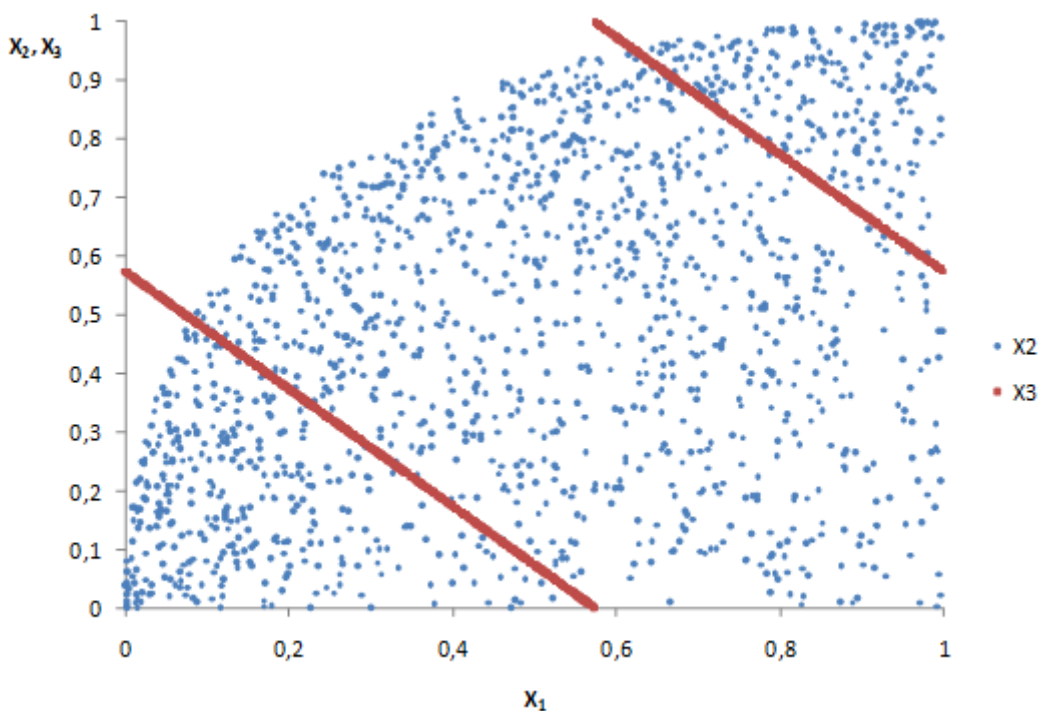
$$\rho(X, Y) = \begin{pmatrix} \rho(X_1, Y_1) & \cdots & \rho(X_1, Y_n) \\ \vdots & \ddots & \vdots \\ \rho(X_n, Y_1) & \cdots & \rho(X_n, Y_n) \end{pmatrix},$$

i. e.  $\rho(X, Y)_{a,b} = \rho(X_a, Y_b)$ ,  $1 \leq a, b \leq n$

**Formula 2: correlation matrix**

The linear correlation coefficient (also called Pearson's linear correlation) measures only *linear* stochastic dependency of two random variables. It takes values between -1 and 1, i. e.  $-1 \leq \rho(X, Y) \leq +1$ . However, perfectly positive correlated random variables do not necessarily feature a linear correlation coefficient of 1 and perfectly negative correlated random variables do not necessarily feature a linear correlation coefficient of -1. Random variables that are strongly dependent may also feature a linear correlation coefficient which is according to amount close to 0.

Furthermore, linear correlation does not provide information of differences regarding the strength of dependencies across the range of values of the random variables. That means that two pairs of random variables with a certain linear correlation coefficient may actually have a completely different dependence structure. The following Figure 1, which shows realizations of two pairs of random variables ( $X_1$  and  $X_2$  respectively  $X_1$  and  $X_3$ ) that both have the same linear correlation, clearly illustrates this:



**Figure 1: dependence structure between random variables  $X_1$  and  $X_2$  respectively  $X_1$  and  $X_3$ <sup>iv</sup>**

The covariance between independent random variables is zero and therefore also the linear correlation. The reversal of this statement does not hold: If the linear correlation coefficient between two random variables is near zero, it can actually exist a high correlation between them.<sup>v</sup>

Linear correlation is a natural dependency ratio for elliptically distributed risks. However, linear correlation can lead to wrong results if used for random variables that are not distributed elliptically.<sup>vi</sup> Especially extreme events with high losses (e. g. catastrophes) can be severely underestimated by using the linear correlation as a measure for dependencies between risks.<sup>vii</sup> Furthermore, for modeling major claims one may have to use distributions with infinite variances. As the linear correlation coefficient is only defined for distributions with finite variances, it is not possible to use it in this case. In addition linear correlation is not invariant concerning non-linear (but monotone) transformation which may cause problems when an amount of loss is converted into a loss payment.<sup>viii</sup>

There are other ratios that can be used for measuring risk dependencies. Two examples are Spearman's rank correlation and Kendall's  $\tau$ . These do not show some of the disadvantages that have been mentioned for linear correlations. However, they do also not provide full information on dependencies between risks, but compress all information into a single number. A further ratio, especially of importance for non-life insurances modeling extreme events, is the coefficient of tail dependence. The coefficient of tail dependence for two random variables  $X$  and  $Y$  describes the likelihood of  $Y$  taking an extreme value on condition that  $X$  also takes an extreme value. This description of the coefficient of tail dependence also highlights that it does not provide *full* information of the dependence structure between random variables. The following Formula 3, Formula 4 and Formula 5 show how the three dependency ratios described in this passage are defined:

$$\rho_S(X, Y) = \rho(F_X(X), F_Y(Y))$$

**Formula 3: Spearman's rank correlation for two continuous random variables**

$$\tau(X, Y) = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$$

**Formula 4: Kendall's  $\tau$**

$$\lambda(X, Y) = \lim_{\alpha \rightarrow 1^-} P(Y > F_Y^{-1}(\alpha) \mid X > F_X^{-1}(\alpha)),$$

on condition that this limit exists.

**Formula 5: coefficient of tail dependence**

### 3. Copulas

In contrast, copulas provide full information on the dependency structure between risks. The concept of copulas is based on separating the joint marginal distribution function into a part that describes the dependence structure and multiple parts that describe the marginal distribution functions.

The Copula was defined by *Sklar*.<sup>ix</sup> The copula itself is a multivariate distribution function with margins that are uniformly distributed on  $[0,1]$ :

$$C(u_1, \dots, u_n) = P(U_1 \leq u_1, \dots, U_n \leq u_n),$$

where  $C(\cdot)$  is the copula,

$(U_1, \dots, U_n)^T$  with  $U_i \sim U(0,1)$  for all  $i = 1, \dots, n$  a vector of random variables  
and  $(u_1, \dots, u_n)^T \in [0,1]^n$  realizations of  $(U_1, \dots, U_n)^T$ .

**Formula 6: copula**

An alternative definition for copulas is given by any function  $C: [0,1]^n \rightarrow [0,1]$  that features the following three properties:

1.  $C(u_1, \dots, u_n)$  is increasing in each component  $u_i$  with  $i \in \{1, \dots, n\}$ .
2.  $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$  for all  $i \in \{1, \dots, n\}$ ,  $u_i \in [0,1]$ .
3. For all  $(a_1, \dots, a_n), (b_1, \dots, b_n) \in [0,1]^n$  with  $a_i \leq b_i$

$$\sum_{i_1=1}^2 \dots \sum_{i_n=1}^2 (-1)^{i_1 + \dots + i_n} C(u_{1i_1}, \dots, u_{ni_n}) \geq 0 \text{ applies,}$$

$$u_{j1} = a_j \text{ and } u_{j2} = b_j \text{ for all } j \in \{1, \dots, n\}.$$

These definitions can be shown to be equivalent.<sup>x</sup>

Using copulas, the risk modeling process consists of two steps. In a first step one has to determine the marginal distribution of every single risk component. Goal of the second step is then to determine the dependence structure between these risk components via the copula function. To obtain the joint distribution function the  $n$  single risks  $X_i$  have then to be transformed each into a random variable  $U_i$  that is uniformly distributed on  $[0,1]$ . This can be achieved using the corresponding marginal distribution  $F_i$ .

$$U_i = F_i(X_i)$$

**Formula 7: transformation of any single risk into a random variable that is uniformly distributed on [0,1]**

We obtain the multivariate distribution function by inserting these transformed random variables into the copula function:

$$F(x_1, \dots, x_n) = C(u_1, \dots, u_n) = C(F_1(x_1), \dots, F_n(x_n))$$

**Formula 8: multivariate distribution function via copula**

In case of continuous and differentiable marginal distributions and a differentiable copula the joint density is:

$$f(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n) c(F_1(x_1), \dots, F_n(x_n)),$$

where  $f_i(x_i)$  is the respective density for distribution function  $F_i$   
and  $c(u_1, \dots, u_n) = \frac{\partial^n C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n}$  the density of the copula.

**Formula 9: density of the multivariate joint distribution derived via copula**

In this way we can derive a multivariate distribution function out of specified marginal distributions and a copula that contains information about the dependence structure between the single variables. Also the opposite holds: A copula can be determined out of the inverse of the marginal distributions and the multivariate distribution function.<sup>x1</sup>

In the following we will describe some of the most important copula families:

- Elliptical copulas
  - Gaussian copulas
  - Student copulas
- Archimedean copulas
  - Gumbel copulas
  - Cook-Johnson copulas
  - Frank copulas

The main differences in these types of copulas are the bands of the resulting distribution in which the dependencies are stronger or weaker.<sup>xii</sup>

Gaussian copulas are also called normal copulas since they lead to multivariate normal distribution functions if standard normal distributions are chosen as marginal distributions. Formula 10 shows the definition of the Gaussian copula:

$$C_{\rho}^{Gau}(u_1, \dots, u_n) = \Phi_{\rho}^n(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)),$$

$\Phi_{\rho}^n$  is the distribution function of the  $n$ -variate standard normal distribution with correlation  $\rho$  and  $\Phi^{-1}$  is the inverse of the distribution function of the univariate standard normal distribution

**Formula 10: Gaussian copula**

The dependency in the tails of multivariate distributions with a Gaussian copula goes to zero<sup>xiii</sup>, which means that the single distribution variables of the joint distribution function are almost independent in case of high realizations even if there is a strong overall correlation between them. Modeling insurance risks means in most cases modeling risks that are independent for lower values but strongly dependent for higher realizations. From this perspective the Gaussian copula does not provide a proper basis for modeling insurance risks.

In contrast, the Student copulas do not feature independency in the tails of a distribution.<sup>xiv</sup> They can be defined by the following Formula 11:

$$C_{v,\rho}^{Stu}(u_1, \dots, u_n) = t_{v,\rho}^n(t_v^{-1}(u_1), \dots, t_v^{-1}(u_n)),$$

$v$  is the number of degrees of freedom,

$t_{v,\rho}^n$  the distribution function of the  $n$ -variate Student distribution

with  $v$  degrees of freedom and a correlation of  $\rho$  and

$t_v^{-1}$  the inverse of the distribution function of the univariate

Student distribution with  $v$  degrees of freedom.

**Formula 11: Student copula**

In the bivariate case Student copula feature asymptotical dependency as long as the correlation between the two random variables is higher than -1 (even if the correlation  $\rho$  is 0 or negative). The strength of the dependence in the tail of the distribution increases with decreasing degrees of freedom  $v$  and with an increasing correlation  $\rho$ .<sup>xv</sup>

The following figures show the densities for both the bivariate Gaussian and the bivariate Student copula.

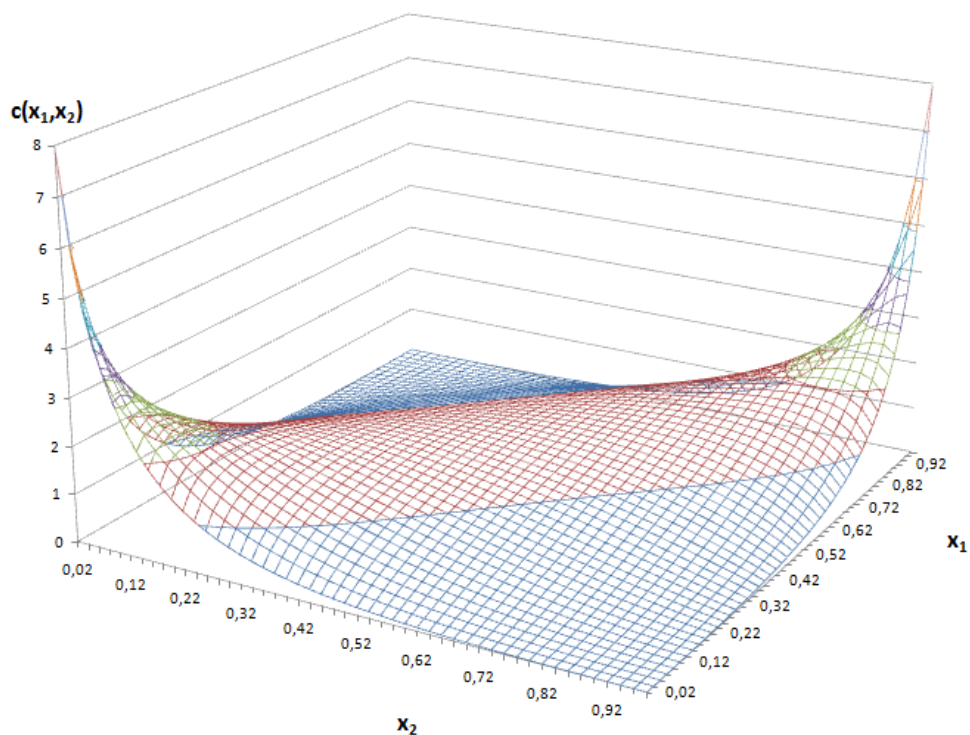


Figure 2: density of a Gaussian copula (correlation  $\rho$  von 0.7)

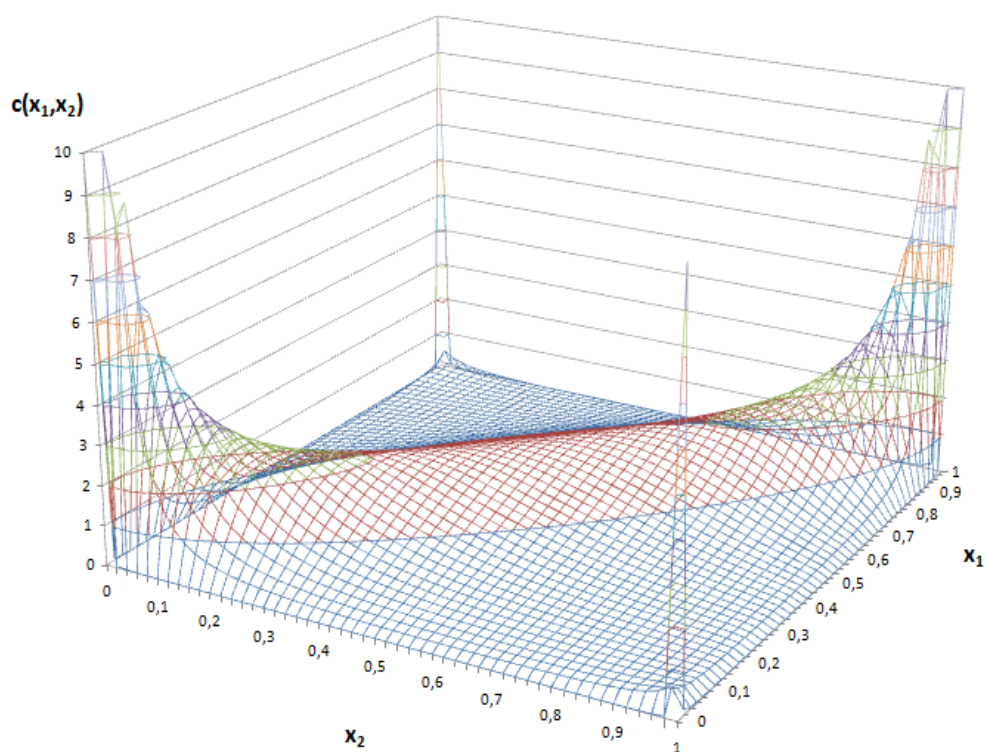


Figure 3: density of a Student copula (correlation  $\rho$  von 0.7)

Another class of copulas is given by the Archimedean copulas to which among others belong the Gumbel copulas. Similarly to the Student copulas they are tail dependent, however not in both the upper and the lower tail, but only in the upper one (see Figure 4). In the lower tail they feature independency. Therefore they are adequate for modeling extreme events: On the one hand stress scenarios<sup>xvi</sup> (with high losses) and high dependence can be captured and on

the other hand common (lower) losses which in general appear independent can be modeled. Formula 12 shows how Gumbel copulas can be defined.<sup>xvii</sup>

$$C_{\beta}^{Gum}(u_1, \dots, u_n) = e^{-(\sum_{i=1}^n (-\ln(u_i))^{\beta})^{\frac{1}{\beta}}},$$

$\beta \geq 1$  is a structural parameter.

**Formula 12: Gumbel copula**

Another Archimedean copula family is represented by the Cook-Johnson copulas.<sup>xviii</sup> Contrary to the Gumbel copulas they are tail dependent only in the lower tail (see Figure 5). Therefore they perform good results if used for modeling yields on shares.<sup>xix</sup> The following formula describes the Cook-Johnson copulas:

$$C_{\beta}^{C-J}(u_1, \dots, u_n) = (u_1^{-\beta} + \dots + u_n^{-\beta} - n + 1)^{\frac{1}{\beta}},$$

$\beta > 0$  is a structural parameter.

**Formula 13: Cook-Johnson copula**

The third type of Archimedean copulas presented in this paper is the Frank copula.<sup>xx</sup> This type of copulas features tail dependence neither in the upper tail nor in the lower tail. The dependence structure given by a copula of this type is similar to one represented by a Normal copula even though the dependence in the tail is even lower (see Figure 6). Formula 14 shows the definition of Frank copulas:

$$C_{\beta}^{Fra}(u_1, \dots, u_n) = -\frac{1}{\beta} \ln\left(1 + \frac{(e^{-\beta u_1} - 1) \dots (e^{-\beta u_n} - 1)}{(e^{-\beta} - 1)^{n-1}}\right),$$

$\beta > 0$  is a structural parameter.

**Formula 14: Frank copula**

The following figures show bivariate examples for densities for each of the three types of Archimedean copulas presented in this paper:

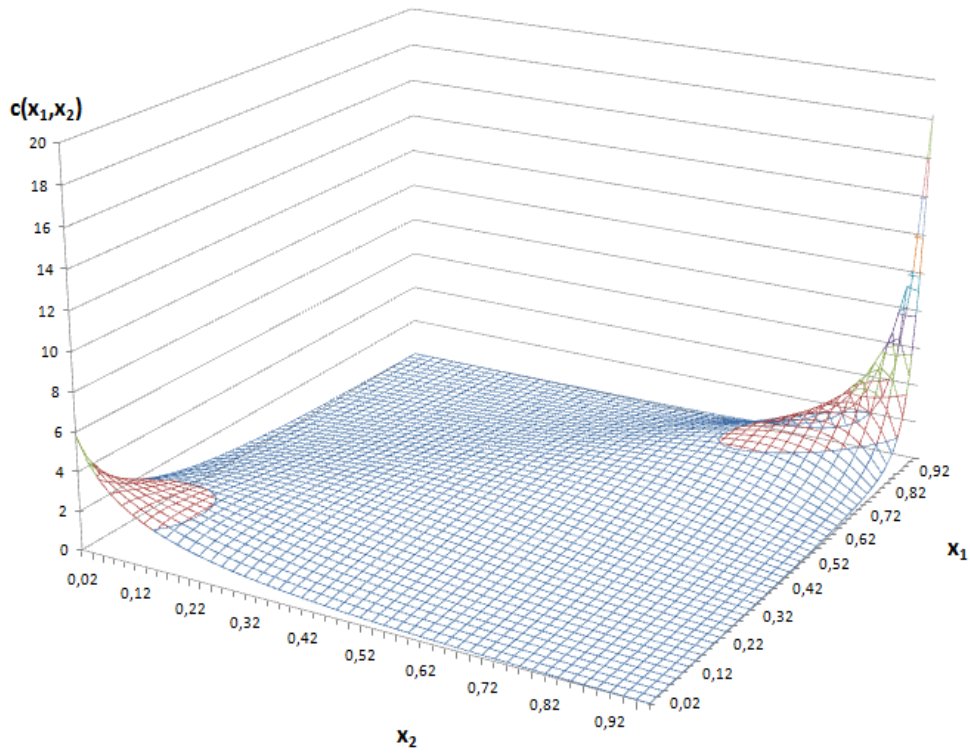


Figure 4: Gumbel copula with structural parameter  $\beta = 2$

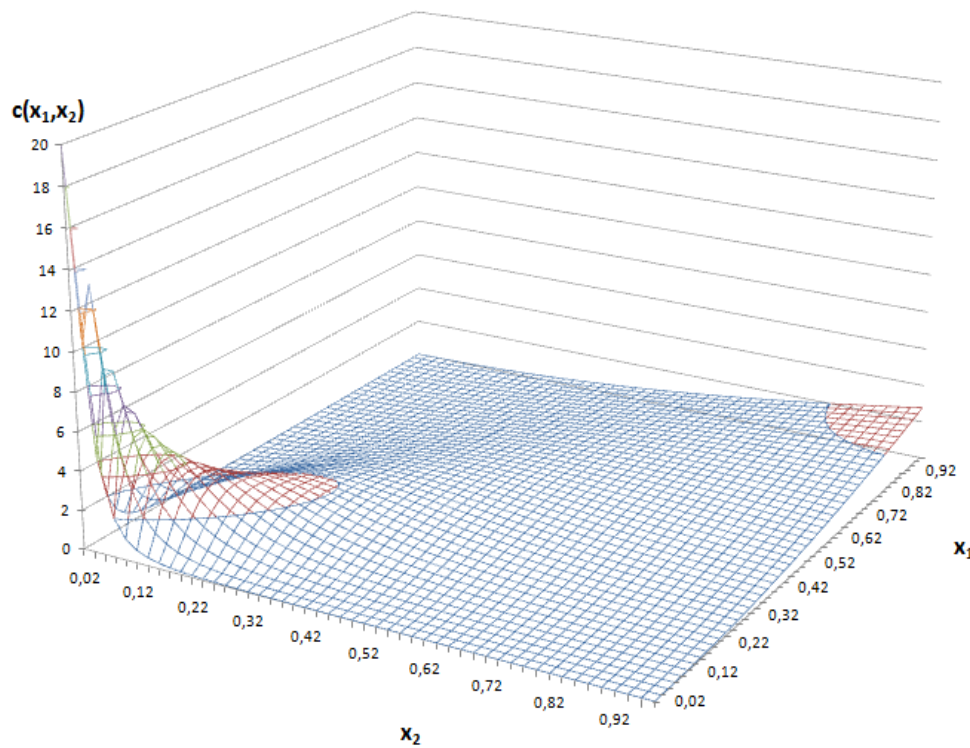


Figure 5: Cook-Johnson copula with structural parameter  $\beta = 2$

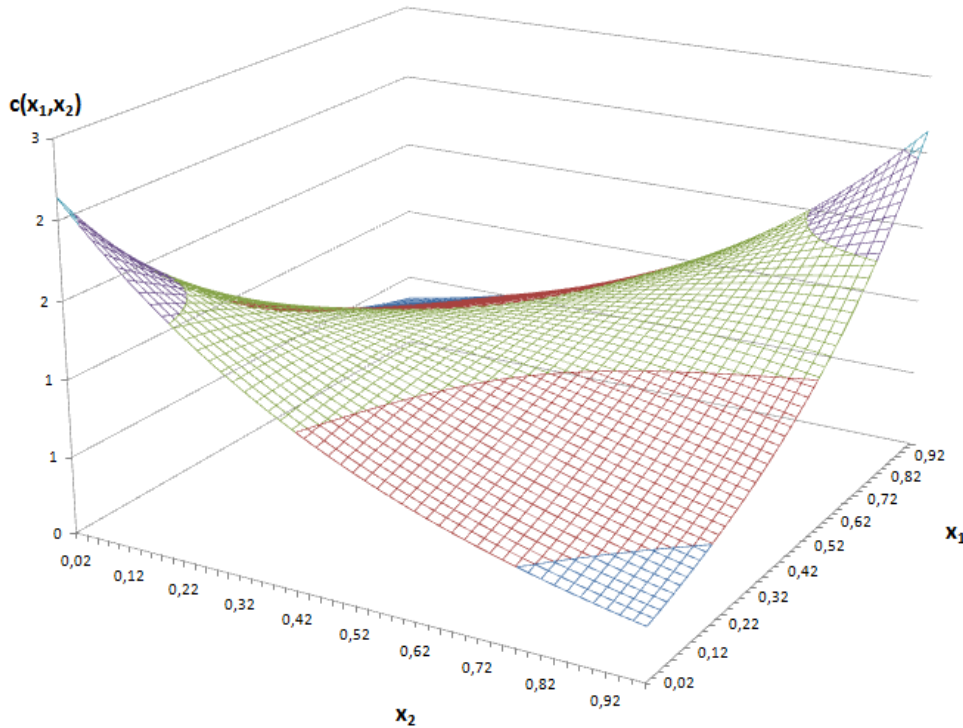


Figure 6: Frank copula with structural parameter  $\beta = 2$

#### 4. Determination of copulas and multivariate distributions

Using the concept of copulas for capturing dependencies between risks in an insurance company, first of all the corresponding copulas have to be determined. Two alternative approaches for achieving this are parametric and non-parametric approaches. Using a parametric approach means to first determine the respective type of copula.<sup>xxi</sup> As shown above the various types of copulas describe a different type of dependence structure each. Therefore it is necessary to choose that type of copulas that best fits the actual dependence structure. We can follow a procedure for the bivariate case established by *Genest, C./Rivest, L.-P. (1993)* which uses the dependency ratios for identifying a type of Archimedean copula that fits the observations. The procedure is carried out in 3 steps:<sup>xxii</sup>

1. Estimation of Kendall's  $\tau$  out of the observations  $(X_{11}, X_{21}), \dots, (X_{1n}, X_{2n})$ .
2. Define an intermediate random variable  $Z_i = F(X_{1i}, X_{2i})$  with distribution function  $K$ . *Genest, C./Rivest, L.-P. (1993)* showed that the following statement holds:

$$K(z) = z - \frac{\phi(z)}{\phi'(z)}$$

$\phi(z)$  is the generator of (and therefore determines) an Archimedean copula.

**Formula 15: relation between distribution function and generator of Archimedean copula**

Construct a non-parametric estimate of  $K$ :

- a. Define pseudo observations  $Z_i = \{ \text{number of } (X_{1j}, X_{2j}) \text{ such that } X_{1j} < X_{1i} \text{ and } X_{2j} < X_{2i} \} / (n - 1)$  for  $i = 1, \dots, n$

- b. Estimate of  $K$  is  $K_n(z) = \{\text{number of } Z_i \leq z\} / \{\text{number of } Z_i\}$ .
3. Construct parametric estimate of  $K$  using the relationship of Formula 15: Use the estimate of Kendall's  $\tau$  from step 1 and the given relation between Kendall's  $\tau$  and the generator of a specific type of a copula  $\phi(z)$  to come to a parametric estimate of  $K$ .

Repeat step three with generators for different types of copulas. At the end choose that type of copula where the parametric estimate of  $K$  most closely resembles the non-parametric estimate of  $K$  calculated in step 2.

Once the type of copula is chosen, the parameters of the copula have to be estimated in such a way that the dependence structure given by the copula fits to the observations at the best. This can be achieved in course of the estimation of the parameters of the marginal distribution by using the maximum likelihood method.<sup>xxiii</sup> Since one of the advantages of using copulas is the separate estimation of the marginal distributions and the dependence structure also a two-step approach can be applied: in the first step the parameters of the marginal distributions are estimated and in a second step those of the copula.<sup>xxiv</sup>

Using the non-parametric approach means determining an empirical copula out of the empirical data and therefore not determining a specific copula type in advance.<sup>xxv</sup> In the same way in which the actual multivariate distribution function is determined by the actual copula and the actual marginal distribution functions, the empirical multivariate distribution function is determined by the empirical copula function and the empirical marginal distribution functions. Furthermore it is given that the empirical copula converges to the actual copula as the number of observations increases.<sup>xxvi</sup>

The requisites of multivariate modeling are similar to those of univariate modeling.<sup>xxvii</sup> However, in contrast to univariate modeling only little experience could be gained in multivariate modeling. Furthermore, the estimation of a copula is highly dependent on the result of the estimation of the marginal distribution functions.<sup>xxviii</sup> Taking a parametric approach this could be avoided by passing onto a semi-parametric approach. This means that the copula itself is determined using a parametric approach, whereas the marginal distribution functions used to do this are non-parametric, empirical functions.<sup>xxix</sup>

## 5. Assessment

In the previous sections we provided an overview of two – very different – concepts for describing dependencies between risks. On the one hand the linear correlation coefficient, Spearman's rank correlation and Kendall's  $\tau$  describe dependencies between risks by a single ratio, on the other hand copulas can be used to build multivariate distributions that capture the whole dependence structure. Both of these concepts will be assessed in the following. However, first we will introduce 5 criteria that dependency ratios should meet.

### a. Criteria for dependency ratios

The following five criteria are desirable for a dependency ratio. Therefore we will first explain the criteria and afterwards match the introduced dependency ratios with them. If  $\delta(\ )$  is a dependency ratio, the criteria can be described as the following:

1. symmetry:  $\delta(X,Y) = \delta(Y,X)$
2. standardization:  $-1 \leq \delta(X,Y) \leq 1$
3. conclusion based on and on co- and countermonotony
  - a.  $\delta(X,Y) = 1 \Leftrightarrow X,Y$  are comonotone
  - b.  $\delta(X,Y) = -1 \Leftrightarrow X,Y$  are countermonotone
4. Invariance with regard to strictly monotone transformations: For a transformation  $T: \mathfrak{R} \rightarrow \mathfrak{R}$  strictly monotone on the codomain of  $X$  the following holds:
  - a.  $\delta(T(X),Y) = \delta(X,Y)$ , if  $T$  is strictly monotonic increasing
  - b.  $\delta(T(X),Y) = -\delta(X,Y)$ , if  $T$  is strictly monotonic decreasing
5. conclusion based on and on independence
 

$\delta(X,Y) = 0 \Leftrightarrow X,Y$  are independent

However, a dependency ratio can never fulfill all of those criteria, as criterion 4 contradicts criterion 5 and the other way around.<sup>xxx</sup>

The first criterion is desirable for dependency ratios because otherwise the resulting dependency ratio would depend on the order of the considered risks. If a dependency ratio fulfils the second criterion, this will lead to a unique measure which makes dependencies between pairs of random variables comparable. Conclusion based on and on co- and countermonotony helps to immediately detect strongly dependent random variables.

Invariance with regard to strictly monotone transformations is mainly important if the dependency ratio is used for practical applications. If a random variable  $X$  is transformed into another variable  $T(X)$  using a strictly monotone function  $T$ , the dependence structure between  $X$  and a second random variable  $Y$  will be the same as the dependence structure between  $T(X)$  and  $Y$ . Therefore also the dependency ratio should take on the same value for  $T(X)$  and  $Y$  as for  $X$  and  $Y$ . The last criterion makes sure that also independency between random variables can be detected.

## **b. Assessment of the introduced concepts**

First, we want to assess the dependency ratios. The most popular of those – the Pearson linear correlation coefficient – only fulfils the first two criteria of the above mentioned five.<sup>xxx</sup> From this point of view it is inferior compared to Spearman's rank correlation and Kendall's  $\tau$  which fulfill the first four of the mentioned five criteria. Furthermore, the Pearson linear correlation coefficient is defined only if the variances of the random variables are finite. Another advantage of both Spearman's rank correlation and Kendall's  $\tau$  is that they do not only measure the linear dependency between random variables, but also the monotone dependence in common.<sup>xxxii</sup> Their calculation may be sometimes easier, but sometimes more difficult than the calculation of the Pearson linear correlation coefficient.<sup>xxxiii</sup>

The coefficient of tail dependence introduced in section 2 should not be compared to one of the above mentioned three dependency ratios, since it focuses only on the dependency in the tails of a distribution. It should therefore be applied if it is required by the respective problem. This is the case mainly if extreme events are modeled. Therefore we think that matching the five criteria with the coefficient of tail dependence is not reasonable.

However, copulas can be used to model multivariate distributions which – in contrast to the above mentioned dependency ratios that compress the whole dependence structure into a single ratio – fully describe the dependence structure. In this way a whole picture of the risks in an insurance company and their dependencies can be provided, whereas the solely use of a dependency ratio reduces the dependencies between risks into a single number and valuable information might get lost. The fact that a given copula implies a certain value for the correlation, but in general not the other way around,<sup>xxxiv</sup> also makes clear that a copula contains much more information than a dependency ratio. Particularly, if we have multivariate distributions where the dependencies are not linear, but are located in the tails, risk could be significantly underestimated if f. ex. the linear correlation coefficient is used.

From a technical point of view copulas offer the opportunity to first model the marginal distribution functions representing the single risks and considering the respective conditions separately and in a second step modeling the dependence structure independently from the single risks. Furthermore, similar to Spearman's rank correlation and Kendall's  $\tau$  copulas are invariant with regard to strictly monotone transformations.<sup>xxxv</sup> An advantage of using copulas instead of directly modeling multivariate distributions is that the marginal distribution function can be of any type, whereas if the multivariate distribution function was directly modeled, each of the marginal distribution functions would have to be of the same type. Besides, directly modeling the multivariate distribution function presumes that a dependency ratio is given.

All in all, we can say that the concept of copulas is clearly superior with regard to the quality of estimating dependencies between risks. Dependency ratios have advantages in their practical usage as they are much easier to calculate, require less effort and much more experience exists with regard to their application in insurance companies. However, if the usage of copulas becomes more popular in future, these advantages of dependency ratios will be likely to disappear.

Copulas do not provide information on whether the random variables are dependent one on another each or only one affects the others. Therefore copulas may indeed fully describe dependencies between risks, however, they do not help us to understand cause-and-effect chains. Further problems of the application of copulas can be seen in the high amount of data required for modeling – especially complex types of – copulas. Particularly for the tails of distributions such observations are not available in sufficient quantities.

However, all in all we can say that since dependency ratios do not provide a complete picture of the actual situation with regard to risk dependencies, therefore provide significant less information and finally may lead to an underestimation of the actual risk of an insurance company, copulas should be used to describe dependencies between risks in an insurance company if possible. Particularly for the option of introducing an internal risk model the application of copulas seems to be suitable.

## 6. Solvency II

### a. Consideration of risk dependencies in Solvency II

According to the new European solvency system (Solvency II) insurance undertakings will have to determine the so called Solvency Capital Requirement (SCR) which reflects the amount of capital that is necessary to limit the probability of ruin to 0.5 %. That implies that

they will also have to determine their *overall* loss distribution function. Hereby at least the following risks have to be considered:<sup>xxxvi</sup>

- non-life underwriting risk
- life underwriting risk
- health underwriting risk
- market risk
- credit risk
- operational risk

Insurers will be able to either use a standard approach or to determine the Solvency Capital Requirement or parts thereof by the use of an internal model. In the latter case the internal model has to be approved by the supervisory authorities. Using the standard approach, the SCR is the sum of the Basic Solvency Capital Requirement, the capital requirement for operational risk and the adjustment for the loss-absorbing capacity of technical provisions and deferred taxes.

The Basic Solvency Capital Requirement consists at least of a risk module for non-life underwriting risk, for life underwriting risk, for health underwriting risk, for market risk and for counterparty default risk each.<sup>xxxvii</sup> After having been determined, the risk modules have to be aggregated. The Solvency II directive clearly states that for the standard approach this has to be done by using correlations (see Table 1) and the following Formula 16:<sup>xxxviii</sup>

$$BSCR = \sqrt{\sum_{i,j} \rho_{i,j} * SCR_i * SCR_j},$$

*BSCR* is the Basic Solvency Capital Requirement,

*SCR<sub>i</sub>* respectively *SCR<sub>j</sub>* are risk-modules *i* respectively *j* and

$\rho_{i,j}$  is the correlation between them.<sup>xxxix</sup>

**Formula 16: aggregation of risk modules in the Solvency II standard approach**

$\rho_{i,j}$	j=1: market risk	j=2: counterparty default risk	j=3: life underwriting risk	j=4: health underwriting risk	j=5: non-life underwriting risk
i=1: market risk	1	0.25	0.25	0.25	0.25
i=2: counterparty default risk	0.25	1	0.25	0.25	0.5
i=3: life underwriting risk	0.25	0.25	1	0.25	0
i=4: health underwriting risk	0.25	0.25	0.25	1	0
i=5: non-life underwriting risk	0.25	0.5	0	0	1

**Table 1: correlation matrix for aggregating risk modules in Solvency II**

## b. Assessment of the Solvency II rules with regard to risk dependencies

On a first level the Basic Solvency Capital Requirement, the capital requirement for operational risk and the adjustment for the loss-absorbing capacity of technical provisions and deferred taxes have to be aggregated. This is done simply by adding the capital requirements which assumes that those risks are fully dependent. However, the assumption

that full dependence between e. g. the operational risk and the risks covered by the BSCR is not realistic and therefore the result for the SCR will be too high.<sup>xi</sup> At the moment there is not sufficient data for a reliable estimation of the operational risk. Against this background it would be sensible to only consider the operational risk qualitatively like Switzerland has decided in the Swiss Solvency Test instead of simply adding an amount of capital to the Basic Solvency Capital Requirement.

In contrast to other solvency systems that are currently in place, in the Solvency II framework dependencies are closely recognized at least in the calculation of the BSCR. However, the given values for the correlations which are shown in Table 1 seem to be highhanded and do not reflect the specific situation of an insurance company. Moreover, we have shown in section 5 that some serious problems may appear if *linear* correlations are used for measuring dependencies and that other dependency ratios should be preferred.

However, insurance companies are able to apply a more precise and sophisticated way of capturing dependencies if they use an internal model. In this case the company can decide to measure dependencies between risks by the use of copulas or at least by the use of other dependency ratios than the linear correlation coefficient. The supervisory authorities may even require the companies to apply an internal model for calculating the Solvency Capital Requirement, or a part thereof, if it is inappropriate to calculate the Solvency Capital Requirement using the standard approach.<sup>xlii</sup> That means that if the approach for considering dependencies that is given in the standard model does not lead to a realistic picture of the actual risk situation of the company, the supervisory authorities may oblige the company to use a more sophisticated way for capturing dependencies. The problem for the supervisory authorities thereby is to obtain an indication that it is not appropriate for a specific insurance company to use the standard approach.

For these reasons we recommend that the Solvency II framework should reward insurers which measure the dependencies between their risks in a more sophisticated way by reducing the SCR or the other way around by imposing higher requirements on companies which use the rudimental standard approach. It would also make sense and give additional incentives to explicitly mention the concept of copulas in the directive. Moreover, we have discovered that the given correlations do not seem to reflect an actual average of the insurance industry. So, if correlations are used, they should at least be actually measured in the insurance industry.

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i See *European Commission (Publ.)* (2009).

ii See *Embrechts, P./McNeil, A./Straumann, D.* (2002), p. 1.

iii See *Tang, A./Valdez, E. A.* (2006), p. 1.

iv See *Pfeifer, D.* (2003), p. 677.

v See *Mummenhoff, A.* (2007), p. 35.

vi See *Embrechts, P./McNeil, A./Straumann, D.* (2002), p. 1.

vii See *Szegö, G.* (2002), pp. 1254-1255.

viii See *Embrechts, P./McNeil, A./Straumann, D.* (2002), pp. 7-8.

ix See *Sklar, M.* (1959).

x See *Embrechts, P./McNeil, A./Straumann, D.* (2002), p. 4.

xi See f. ex. *Faivre, F.* (2003), p. 5 or *Mummenhoff, A.* (2007), pp. 38-39.

xii See *Venter, G. G.* (2002), p. 68.

xiii See f. ex. *Embrechts, P./McNeil, A./Straumann, D.* (2002), p. 19 or *Tang, A./Valdez, E. A.* (2006), p. 6. This does not hold if the correlation equals 1 or -1.

xiv See f. ex. *Faivre, F.* (2003), p. 6 oder *Tang, A./Valdez, E. A.* (2006), p. 6.

xv If the number of degrees of freedom is  $\nu = 1$ , we obtain the so called Cauchy copulas. See *Tang, A./Valdez, E. A.* (2006), p. 6.

xvi We have a stress scenario if the actual situation deviates highly to the disadvantage of the insurer compared to the expected situation. See *Zwiesler, H.-J.* (2005), pp. 125-126.

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- xvii  $\beta = 1$  leads to a multivariate distribution of independent random variables. Only in this case the Gumbel copula is independent in the upper tail.
- xviii In the literature the Cook-Johnson copulas are also called Pareto copulas or Clayton copulas. See *Nelsen, R. B. (2006)*, p. 118 and *Tang, A./Valdez, E. A. (2006)*, p. 6.
- xix See *Ané, T./Kharoubi, C. (2003)*, p. 429.
- xx See *Ané, T./Kharoubi, C. (2003)*, p. 417, *Junker, M./May, A. (2005)*, p. 432 and *Venter, G. G. (2002)*, pp. 84-85.
- xxi See *Szegö, G. (2002)*, p. 1268.
- xxii See *Frees, E. W./Valdez, E. A. (1998)*, pp. 9-10.
- xxiii See *Junker, M./May, A. (2005)*, p. 437.
- xxiv See *Junker, M./May, A. (2005)*, p. 437.
- xxv See *Szegö, G. (2002)*, p. 1268.
- xxvi See *Ané, T./Kharoubi, C. (2003)*, p. 425.
- xxvii See *Pfeifer, D. (2003)*, p. 681.
- xxviii See *Pfeifer, D. (2003)*, p. 681.
- xxix See *Junker, M./May, A. (2005)*, p. 437. This approach is also used for identifying a specific Cook-Johnson copula as the parametric copula that best describes the dependencies between the yields of different stock indexes. See *Ané, T./Kharoubi, C. (2003)*, pp. 424-431.
- xxx For a proof see *Embrechts, P./McNeil, A./Straumann, D. (2002)*, p. 15.
- xxxi See *Embrechts, P./McNeil, A./Straumann, D. (2002)*, p. 15.
- xxxii See *Embrechts, P./McNeil, A./Straumann, D. (2002)*, p. 16.
- xxxiii Using f. ex. multivariate normal distributions or multivariate Student distributions the calculation of the momentum based linear correlation coefficient is easier. However, if we consider multivariate distributions that have a dependence structure represented by a Gumbel copula, the calculation of Spearman's rank correlation and Kendall's  $\tau$  might be easier.
- xxxiv See f. ex. Figure 1.
- xxxv See *Pfeifer, D. (2003)*, p. 679.
- xxxvi See *European Commission (Publ.) (2009)*, Article 101, Section 4.
- xxxvii See *European Commission (Publ.) (2009)*, Article 104, Section 1. These risk modules have to be split into sub-modules. See *European Commission (Publ.) (2009)*, Article 105. The sub-modules shall be aggregated using the same approach as for the aggregation of the risk modules that is described in the following.
- xxxviii See *European Commission (Publ.) (2009)*, Annex IV, Item 1.
- xxxix  $i,j=1$ : risk module for market risk,  $i,j=2$ : risk module for counterparty default risk,  $i,j=3$ : risk module for life underwriting risk,  $i,j=4$ : risk module for health underwriting risk,  $i,j=5$ : risk module for non-life underwriting risk.
- xl However, the amount of solvency capital for the operational risk is limited. See *European Commission (Publ.) (2009)*, Article 106, Section 3.
- xli See *European Commission (Publ.) (2009)*, Article 117.

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